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# HOCHSCHILD HOMOLOGY CRITERIA FOR TRIVIAL ALGEBRA STRUCTURES

MICHELINE VIGUÉ-POIRRIER

ABSTRACT. We prove two similar results by quite different methods. The first one deals with augmented artinian algebras over a field: we characterize the trivial algebra structure on the augmentation ideal in terms of the maximality of the dimensions of the Hochschild homology (or cyclic homology) groups. For the second result, let $X$ be a 1-connected finite CW-complex. We characterize the trivial algebra structure on the cohomology algebra of $X$ with coefficients in a fixed field in terms of the maximality of the Betti numbers of the free loop space.

## INTRODUCTION

Let $A$ be a unital algebra over a field $k$. By definition, the Hochschild homology of $A$ with coefficients in $A$ (called here Hochschild homology) is the homology of the Hochschild complex $(C_*(A), b)$ (see section 1) and is denoted $HH_*(A) = \bigoplus_{n\geq 0} HH_n(A)$, [Lo]. If $A$ is a finite dimensional $k$-vector space, then each homology group is finite dimensional and the $n$-th Betti number of $HH_*(A)$ is defined as the dimension of the $k$-vector space $HH_n(A)$.

In §1, we focus our attention on augmented algebras $A = k \oplus \bar{A}$ where $\bar{A}$ is the augmentation ideal and we assume that $\bar{A}$ is a finite dimensional $k$-vector space. When $\bar{A} \cdot \bar{A} = 0$, the calculation of the Betti numbers of $HH_*(A)$ has been carried out by many people, [L-Q], [Ro], [Pa]. If $A$ is any augmented algebra with augmentation ideal $\bar{A}$, we can consider $A_t = k \oplus \bar{A}_t$ the augmented algebra with augmentation ideal $\bar{A}_t$, where $\bar{A}_t = \bar{A}$ as a $k$-vector space and $\bar{A}_t \cdot \bar{A}_t = 0$. We first show (theorem 1.3) that

(1) $\dim HH_n(A) \leq \dim HH_n(A_t)$ for $n \geq 0$,
(2) $\dim HC_n(A) \leq \dim HC_n(A_t)$ for $n \geq 0$

where $(HC_n(\cdot))_{n\geq 0}$ are the cyclic homology groups ([Lo] and definition below).

Then (theorems 1.4, 1.6) we prove that the inequalities above become equalities if and only if the multiplication is trivial on the augmentation ideal, so that we characterize the trivial algebra structure on $A$ in terms of the maximality of the Betti numbers of the Hochschild homology or cyclic homology groups.

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More precisely, we first show that if (2) is an equality for some $n \geq 1$ and $\bar{A}$ is a nilpotent ideal, then $A = A_t$ as algebras. After that, we show that if (1) is an equality for some $n \geq 1$ and $A$ is a commutative algebra, then $A = A_t$ as algebras.

More generally, if $(A, d)$ is a differential graded algebra over a field $k$, we can define Hochschild and cyclic homology groups as above [Go], [Lo].

Let $X$ be a simply connected space. A result of Burghelea-Fiedorowicz [B-F], or Goodwillie [Go], or Jones [Jo] asserts that

$$H_*(X^{S^1}, k) = HH_*(C_*(\Omega X, k)),$$
$$H_*^{S^1}(X^{S^1}, k) = HC_*(C_*(\Omega X), k)$$

as $k$-graded vector spaces, where $C_*(\Omega X, k)$ is the differential graded algebra of singular chains on the Moore loop space of $X$, and $X^{S^1}$ is the free loop space on $X$, i.e., the space of all continuous maps from the circle into $X$. The group $S^1$ acts on $X^{S^1}$ by rotating the loops, and we denote by $H_*^{S^1}(X^{S^1}, k)$ the homology of the Borel construction associated to this action.

The study of the Betti numbers of the free loop space is motivated by its applications to geometry. A famous example is the Gromoll-Meyer theorem [G-M] which asserts the following: if $X$ is a 1-connected compact manifold such that the sequence of Betti numbers dim $H^n(X^{S^1}, k)$ is not bounded, then there exist infinitely many closed distinct geodesics on $X$ for any Riemannian metric. Roos [Ro] and Parhizgar [Pa] compute explicitly these Betti numbers when $X$ is a wedge of spheres of the same dimension; in this case, $H^*(X, k)$ is an augmented algebra and the multiplication is trivial on $\tilde{H} = \bigoplus_{n>0} H^n(X, k)$.

In §2, we give, in the context of topology, similar results to those of §1. The proofs are quite different and rely on the notion of an Adams-Hilton model for a space [A-H]. Let $H = \bigoplus_{n \geq 0} H_n$ be a graded $k$-vector space with $H_0 = k$, $H_1 = 0$ and $\bar{H} = \bigoplus_{n>0} H_n$ finite dimensional. We consider the wedge of spheres $X_t = \bigvee_{1 \leq i \leq r} S^{n_i}$, where the sequence $(n_1, \ldots, n_r)$ corresponds to a basis of the vector space $\bar{H}$. We have $H^*(X_t, k) = \operatorname{Hom}(H, k)$.

Theorem 2.1 asserts that if $X$ is any finite complex with $H_*(X, k) = H$, then the Betti numbers of the free loop space on $X$ and the equivariant free loop space on $X$ are bounded by those of $X_t$. Conversely, we prove (theorem 2.2 and theorem 2.4) that if the Betti numbers are the same, then the multiplication is trivial on $\tilde{H}^*(X, k) = \bigoplus_{n>0} H^n(X, k)$.

As in the algebraic case, we characterize the trivial algebra structure on $H^*(X, k)$ in terms of the maximality of the Betti numbers of the free loop space.

Recall from [Vi1] that if $X_t = \bigvee_{1 \leq i \leq r} S^{n_i}$, $r \geq 2$, we have proved that the sequence of Betti numbers of the free loop space grows exponentially. When $X$ is a hyperbolic space, many attempts have been made to prove the exponential growth of the Betti numbers, [La1], [La2]. Results in this paper convince us that the proof will be harder when multiplication is not trivial on $\tilde{H}^*(X, k)$.

## 1. The Hochschild and cyclic homology of augmented algebras

In the following, we always consider augmented algebras over a fixed field $k$. We recall, from [Lo], the definitions of Hochschild and cyclic homology.

The Hochschild homology groups $(HH_n(A))_{n\geq 0}$ are the homology groups of the Hochschild complex $(C_*(A), b)$. If $\bar{A}$ is the augmentation ideal of $A$, we define $C_*(A) = \bigoplus_{n\geq 0} C_n(A)$ and $b$ by the following formulas:

$$C_n(A) = A \otimes \bar{A}^{\otimes n},$$

$$b(a_0 \otimes a_1 \otimes \ldots \otimes a_n) = \sum_{i=0}^{n-1} (-1)^i a_0 \otimes \ldots \otimes a_i a_{i+1} \otimes \ldots \otimes a_n + (-1)^n a_n a_0 \otimes a_1 \otimes \ldots \otimes a_{n-1}$$

where $a_0 \in A$, $a_i \in \bar{A}$ if $i \geq 1$.

All the tensor products are over $k$.

We introduce the cyclic permutation $t_n : \bar{A}^{\otimes n} \to \bar{A}^{\otimes n}$ defined by $t_n(a_1 \otimes \ldots \otimes a_n) = (-1)^{n-1}\, a_n \otimes a_1 \otimes \ldots \otimes a_{n-1}$.

From Loday, [Lo], Proposition 2.2.14, the reduced cyclic homology groups $H\tilde{C}_n(A) := HC_n(A)/HC_n(k)$ can be computed as the homology groups of the complex $\bar{A}^{\otimes(n+1)}/(Id - t_{n+1})$ endowed with the differential induced by $b$, when $\operatorname{char} k = 0$, or $\operatorname{char} k = p$ and $n < p - 1$.

**Proposition 1.1** ([Ro])**.** *Let $A$ be an augmented algebra, where the augmentation ideal $\bar{A}$ is finite dimensional and satisfies $\bar{A} \cdot \bar{A} = 0$. Then*

1. $HH_n(A) = \operatorname{Coker}(\mathrm{Id} - t_{n+1}) \oplus \operatorname{Ker}(\mathrm{Id} - t_n)$ *as $k$-vector spaces for all $n > 0$;*
2. *if* $\operatorname{char} k = 0$, $H\tilde{C}_n(A) = \operatorname{Coker}(\mathrm{Id} - t_{n+1})$ *for all $n > 0$.*

**Corollary 1.2.** *Let $A$ be an augmented algebra, where the augmentation ideal $\bar{A}$ is finite dimensional and satisfies $\bar{A} \neq 0$, $\bar{A} \cdot \bar{A} = 0$. Then*

1. $HH_n(A) \neq 0$ *for all $n > 0$;*
2. *if* $\operatorname{char} k = 0$, $H\tilde{C}_{2n}(A) \neq 0$ *for all $n \geq 0$ and $HC_{2n+1}(A) \neq 0$ for all $n \geq 0$ if* $\dim \bar{A} \geq 2$.

*Proof.* Let $x$ be a nonzero element in $A$. Then the element $x \otimes x \otimes \ldots \otimes x$ belongs to $\bar{A}^{\otimes(2n+1)} \cap \operatorname{Ker}(\mathrm{Id} - t_{2n+1})$.

If $\dim \bar{A} \geq 2$, let $x$ and $y$ be two independent elements in the $k$-vector space $\bar{A}$. Put $X = \sum_{i=1}^{2n} X_i$ with $X_i = (-1)^{i-1}\, x \otimes \ldots \otimes y \otimes \ldots \otimes x$ with $y$ at the $i$th place in the tensor product $\bar{A}^{\otimes(2n)}$; we check that $X$ belongs to $\bar{A}^{\otimes(2n)} \cap \operatorname{Ker}(\mathrm{Id} - t_{2n})$ so that $HC_{2n-1}(A) \neq 0$. □

*Remark.* If $\operatorname{char} k = p$, $p > 0$, part 2 of proposition 1.1 and corollary 1.2 remain valid for $n < p - 1$.

**Theorem 1.3.** *Let $A$ be an augmented algebra over a field $k$. Let $\bar{A}$ be its augmentation ideal, with $\dim_k \bar{A}$ finite. Let $A_t = k \oplus \bar{A}_t$ be the augmented algebra with trivial multiplication on $\bar{A}_t$ and $\bar{A} = \bar{A}_t$ as a $k$-vector space. We assume that* $\operatorname{char} k = 0$, *or* $\operatorname{char} k \neq 0$ *and there exists $N \geq 2$ such that $\bar{A}^N = 0$ in $A$; then we have*

1. $\dim HH_n(A) \leq \dim HH_n(A_t)$ *for all* $n \geq 0$*;*
2. $\dim HC_n(A) \leq \dim HC_n(A_t)$ *for all* $n \geq 0$.

*Remark.* If $A$ is local, or if $A = \bigoplus_{n\geq 0} A_n$ is a graded algebra with $A_0 = k$, there exists $N \geq 2$ such that $\bar{A}^N = 0$. Otherwise, it cannot be true; for example, if $A = k[X]/(X^2 - X)$, we have $\bar{A}^N = \bar{A} = (X)$ for all $N \geq 1$. In this case, parts 1) and 2) of the proposition remain true for $n < \operatorname{char} k - 1$.

*Proof.* If $\operatorname{char} k = 0$, the reduced cyclic homology can be computed using the reduced Connes complex $\bar{C}^\lambda_* = \bigoplus_n \bar{C}^\lambda_n$ with $\bar{C}^\lambda_n = \bar{A}^{\otimes n+1}/(Id - t_{n+1})$. We denote by $\bar{b}$ its differential; we have seen that $H\tilde{C}_n(A_t) = \bar{A}^{\otimes n+1}/(Id - t_{n+1})$, and we have $H\tilde{C}_n(A) = (\bar{A}^{\otimes n+1}/(Id - t_{n+1}), \bar{b})$. This implies trivially that $\dim H\tilde{C}_n(A) \leq \dim H\tilde{C}_n(A_t)$. Using the Connes long exact sequence, we see that

$$\dim HH_n(A) \leq \dim H\tilde{C}_{n-1}(A) + \dim H\tilde{C}_n(A) \text{ for } n \geq 1;$$

so we have $\dim HH_n(A) \leq \dim H\tilde{C}_{n-1}(A_t) + \dim H\tilde{C}_n(A_t)$.

$A_t$ can be considered as a graded algebra with $A_0 = k$; so we can use Goodwillie's trick ([Lo], theorem 4.1.13) and we have

$$\dim HH_n(A_t) = \dim H\tilde{C}_{n-1}(A_t) + \dim H\tilde{C}_n(A_t).$$

This gives us $\dim HH_n(A) \leq \dim HH_n(A_t)$ for all $n$.

If $k$ is an arbitrary field, we use a spectral sequence argument to prove theorem 1.3. We define an increasing filtration on the Hochschild complex as follows:

$$\begin{aligned} &\mathcal{F}_{-p}(C_n(A)) = A \otimes (\bar{A}^p)^{\otimes n} \quad &\text{for} \quad p \geq 1, \\ &\mathcal{F}_k(C_n(A)) = C_n(A) \quad &\text{for} \quad k \geq 0. \end{aligned}$$

With the additional hypothesis that there exists $N \geq 2$ such that $\bar{A}^N = 0$, this filtration is bounded and gives rise to a spectral sequence $(E^r_{**}, d^r)$ converging to $HH_*(A)$ with

$$\begin{aligned} &E^0_{-p,n+p} = A \otimes \left(\frac{\bar{A}^p}{\bar{A}^{p+1}}\right)^{\otimes n} \quad \text{for} \quad p \geq 1, \\ &E^0_{kq} = 0 \quad \text{if} \quad k \geq 0. \end{aligned}$$

We check that $d^0((\lambda + \bar{a}_0) \otimes \bar{a}_1 \otimes \ldots \otimes \bar{a}_n) = \lambda(Id - t_n)(\bar{a}_1 \otimes \ldots \otimes \bar{a}_n)$ where $\bar{a}_0 \in \bar{A}$ and $\bar{a}_i \in \bar{A}^p/\bar{A}^{p+1}$ for $i \geq 1$.

So we have $\bigoplus_p E^1_{-p,n+p} = HH_n(A_t)$. This implies

$$\dim HH_n(A) \leq \dim HH_n(A_t).$$

To prove 2), we use the reduced bicomplex $\bar{B}(A)$ to compute cyclic homology ([Lo], page 58), and we define on it an increasing filtration as above.

Now, we are interested in algebras $A$ for which inequality 1 or 2 of theorem 1.3 becomes an equality. □

**Theorem 1.4.** *Let* $A$ *be an augmented algebra over a characteristic zero field* $k$*. Let* $\bar{A}$ *be its augmentation ideal. We assume that* $\bar{A}$ *is a finite dimensional* $k$*-vector space and there exists* $N \geq 2$ *such that* $\bar{A}^N = 0$*, and* $\bar{A} \neq 0$*. Let* $A_t = k \oplus \bar{A}_t$ *be* *the augmented algebra with trivial multiplication on $\bar{A}_t$, and $\bar{A}_t = \bar{A}$ as a $k$-vector space. Suppose that there exists $n \geq 1$ such that*

$$\dim HC_n(A) = \dim HC_n(A_t).$$

*Then the multiplication is trivial in the augmented algebra $A$ (namely, $A$ is isomorphic to $A_t$, as augmented algebras).*

*Proof.* We have $H\tilde{C}_n(A) = H_n(\bar{A}^{\otimes(n+1)}/(Id - t_{n+1}), \bar{b})$. If $\dim HC_n(A) = \dim HC_n(A_t)$, then $\dim H\tilde{C}_n(A) = \dim H\tilde{C}_n(A_t)$. This is equivalent to the facts that $\bar{b} = 0$ is zero on $\bar{A}^{\otimes(n+1)}/(Id - t_{n+1})$ and $\bar{b}$ is zero on $\bar{A}^{\otimes(n+1)}/(Id - t_{n+2})$. This implies that there exists $m \geq 1$ such that

$$b(\bar{A}^{\otimes(2m+1)}) \quad \text{belongs to} \quad (Id - t_{2m})\,(\bar{A}^{\otimes(2m)}).$$

But we have

$$(Id - t_{2m})\,(\bar{A}^{\otimes 2m}) = \mathrm{Ker}(Id + t_{2m} \ldots + t_{2m}^{2m-1}).$$

We can consider $x \otimes \ldots \otimes x$ in $\bar{A}^{\otimes(2m+1)}$ where $x$ is a nonzero element in $\bar{A}$.

We have $b(x \otimes \ldots \otimes x) = 2x^2 \ldots \otimes x + \sum_{i=2}^{2m} (-1)^{i-1} x \otimes \ldots \otimes x^2 \ldots \otimes x$.

We check that $X = (Id + t_{2m} + \ldots + t_{2m}^{2m-1}) \circ b(x \otimes \ldots \otimes x) = \sum_{1 \leq i \leq 2m} X_i$ with $X_i = (-1)^{i-1}(2m+1) x \otimes \ldots \otimes x^2 \otimes \ldots \otimes x$ and $x^2$ at the $i$th place.

Let $(x, e_2, \ldots, e_d)$ be a basis of the $k$-vector space $\bar{A}$. We can write $x^2 = \lambda x + \sum_{2 \leq i \leq d} \lambda_i e_i$ with $\lambda \in k$, $\lambda_i \in k$, $2 \leq i \leq d$.

The condition $X = 0$ implies that $(2m+1)\,\lambda_i = 0$ for $2 \leq i \leq d$. Since $2m+1 < p$, we should have $\lambda_i = 0$ for $2 \leq i \leq d$. The hypothesis $\bar{A}^N = 0$ implies $\lambda = 0$, so that $x^2 = 0$, for any $x \in \bar{A}$.

If $\dim \bar{A} = 1$, we have proved that $\bar{A} \cdot \bar{A} = 0$.

If $\dim \bar{A} \geq 2$, consider two independent elements $x$ and $y$ in $\bar{A}$; we have $0 = (x+y)^2 = x^2 + y^2 + xy + yx$, and so we have

$$xy + yx = 0.$$

From the hypothesis, we have seen that there exists $m \geq 1$ such that $b(\bar{A}^{\otimes 2m})$ belongs to $(Id - t_{2m-1})\,(\bar{A}^{\otimes(2m-1)})$.

Let $X = y \otimes x \otimes \ldots \otimes x$ in $\bar{A}^{\otimes 2m}$,

$$b(X) = (yx - xy) \otimes x \ldots \otimes x - y \otimes x^2 \otimes x \ldots \otimes x \ldots + y \otimes x \ldots \otimes x^2.$$

Since $x^2 = 0$ and $xy = -yx$, we have

$$b(x) = 2yx \otimes x \otimes x \otimes \ldots \otimes x.$$

Let $(x, y, e_3, \ldots, e_d)$ be a basis of the $k$-vector space $\bar{A}$. We can write $yx = \mu x + \nu y + \sum_{i \geq 3} \alpha_i e_i$ with $\mu, \nu, \alpha_i \in k$.

If $b(X)$ belongs to $(Id - t_{2m-1})\,(\bar{A}^{\otimes(2m-1)}) = \mathrm{Ker}(Id + t_{2m-1} + \ldots + t_{2m-1}^{2m-2})$, then we should have $(2m-1)\,\mu = 0$, $2\nu = 0$, $2\alpha_i = 0$. This proves that $\bar{A} \cdot \bar{A} = 0$.

So, $A$ is an augmented commutative algebra isomorphic to $A_t$. □

*Remark.* Theorem 4.1 remains valid if $\mathrm{char}\, k = p$, $p > 0$, and $n < p - 2$.

**Example 1.5.** Let $A = k[X]/(X^2 - X)$ and $A_t = k[X]/X^2$. We check, [B-V], that

$$\begin{aligned} H\tilde{C}_{2n}(A_t) &= H\tilde{C}_{2n}(A) = k, \\ HC_{2n+1}(A_t) &= HC_{2n+1}(A) = 0, \\ HH_n(A_t) &= k \quad \text{for all} \quad n > 0, \\ HH_n(A) &= 0 \quad \text{for all} \quad n > 0. \end{aligned}$$

This shows that the hypothesis $\bar{A}^N = 0$ cannot be omitted in theorem 1.4. On the other hand, the Hochschild homology groups of $A_t$ and $A$ are quite distinct.

This observation leads us to state the following for commutative algebras.

**Theorem 1.6.** *Let $A$ be an augmented commutative algebra over a characteristic zero field $k$. Let $\bar{A}$ be its augmentation ideal and assume that $\bar{A}$ has finite dimension. Let $A_t = k \oplus \bar{A}_t$ be the augmented algebra with trivial multiplication on $\bar{A}_t$ and $\bar{A}_t = \bar{A}$ as a $k$-vector space. Suppose that there exists $n \geq 1$ such that*

$$HH_n(A) = HH_n(A_t).$$

*Then the multiplication is trivial in the augmented algebra $A$ (namely $A$ is isomorphic to $A_t$ as augmented algebras).*

*Proof.* Assume that there exists $n \geq 1$ such that

$$HH_n(A) = HH_n(A_t).$$

From the Connes exact sequence, we see that

$$\dim HH_n(A) \leq \dim H\tilde{C}_n(A) + \dim H\tilde{C}_{n-1}(A).$$

From theorem 1.3, we know that

$$\dim H\tilde{C}_{n-1}(A) + \dim H\tilde{C}_n(A) \leq \dim H\tilde{C}_{n-1}(A_t) + \dim H\tilde{C}_n(A_t).$$

But, if $\dim HH_n(A) = \dim HH_n(A_t)$ for some $n$, we have

$$\dim HH_n(A_t) = \dim H\tilde{C}_n(A_t) + \dim H\tilde{C}_{n-1}(A_t) \leq \dim H\tilde{C}_n(A) + \dim H\tilde{C}_{n-1}(A).$$

The two inequalities imply that

$$\begin{aligned} \dim H\tilde{C}_n(A) &= \dim H\tilde{C}_n(A_t), \\ \dim H\tilde{C}_{n-1}(A) &= \dim H\tilde{C}_{n-1}(A_t). \end{aligned}$$

The first part of the proof of theorem 1.4 implies that, for any nonzero element $x$ in $\bar{A}$, there exists $\lambda \in k$, such that $x^2 = \lambda x$.

Denote by $(x_1, \dots, x_m)$ a $k$-basis of $\bar{A}$; so there exist $\lambda_i \in k$ for $1 \leq i \leq m$ such that $x_i^2 = \lambda_i x_i$. If $m = 1$, we recover example 1.5 and the theorem is true. If $m \geq 2$, consider two independent elements $x_i$, $x_j$, $i \neq j$ in $\bar{A}$. Using the fact that there exist $(\mu, \nu) \in k^2$ such that

$$\begin{aligned} (x_i + x_j)^2 &= \mu(x_i + x_j), \\ (x_i - x_j)^2 &= \nu(x_i - x_j), \end{aligned}$$

we show that

$$x_i x_j = \frac{\lambda_j}{2} x_i + \frac{\lambda_i}{2} x_j.$$

This proves that $A = S/I$ where $S$ is the polynomial algebra $k[X_1, \dots, X_m]$ and $I$ is generated by

$$f_i = X_i^2 - \lambda_i X_i,\ 1 \le i \le m, \qquad g_{ij} = X_i X_j - \frac{\lambda_j}{2} X_i - \frac{\lambda_i}{2} X_j,\ 1 \le i < j \le m.$$

Using the Jacobian criterion, it is easy to check that if there exists $i$ with $\lambda_i \neq 0$, then $A$ is smooth over $k$. Since $A$ is artinian, the module of Kahler differential forms $\Omega^1_{A/k}$ should be zero.

Using the Hochschild-Kostant-Rosenberg theorem [H-K-R], we see that all the Hochschild homology groups $HH_n(A)$ should be zero for $n \ge 1$.

We have seen, in corollary 1.2, that $HH_n(A_t)$ is nonzero for all $n$.

So, if there exists $n \ge 1$ such that $HH_n(A) = HH_n(A_t)$, then $HH_n(A) \neq 0$ and $A$ is not smooth over $k$. This occurs only if $\lambda_i = 0$ for all $i \in \{1, \dots, m\}$. In that case $A = k[X_1, \dots, X_m]/((X_i^2)_{1 \le i \le m}, (X_i X_j)_{1 \le i < j \le m})$. □

*Remark.* Theorem 1.6 remains valid if $\operatorname{char} k = p > 3$ and $1 \le n < p - 1$.

## 2. Characterization of spaces with given homology vector space in terms of the Betti numbers of the free loop space

In the following, we fix an arbitrary field $k$.

Let $X$ be a simply-connected finite $CW$-complex. Denote by $X^{S^1}$ the space of all continuous maps from the circle into $X$. Since the circle acts on $X^{S^1}$ by rotating loops, we can apply to $X^{S^1}$ the Borel construction and we get a space denoted by $ES^1 \times_{S^1} X^{S^1}$ whose homology, denoted $H_*^{S^1}(X^{S^1}, k)$, is called the equivariant homology of the free loop space.

The starting point for our study is a result of Burghelea-Fiedorowicz [B-F], Goodwillie [Go], and J. Jones [Jo], which asserts the following:

$$H_*(X^{S^1}, k) = HH_*(C_*(\Omega X, k)),$$
$$H_*^{S^1}(X^{S^1}, k) = HC_*(C_*(\Omega X, k))$$

where $C_*(\Omega X, k)$ is the differential graded algebra of singular chains on the Moore loop space of $X$.

The definitions of Hochschild and cyclic homology can be extended easily to the category of differential graded chain algebras ([Lo], chapter 5).

Results of §1 lead us to ask the following question, in the topological framework:

*Let $H = \bigoplus_{n \ge 0} H_n$ be a finite dimensional graded vector space over an arbitrary field $k$, with $H_0 = k$, $H_1 = 0$, $\bar{H} = \bigoplus_{n \ge 1} H_n \neq 0$. Let $(e_1, \dots, e_r)$ be a homogeneous basis of $\bar{H}$. Then we put $X_t = \bigvee_{1 \le i \le r} S^{n_i}$, where $n_i$ is the degree of $e_i$. Hence $H_*(X_t, k)$ is canonically isomorphic to $H$. Now, if $X$ is any finite $CW$-complex with $H_*(X, k) = H$, can we compare the Betti numbers of the free loop space on $X$ and on $X_t$, and characterize spaces $X$ such that $X^{S^1}$ and $X_t^{S^1}$ have the same Betti numbers?*

Unfortunately, this problem cannot be tackled as in the algebraic context, because the notion of Connes complex, defined as a quotient of the Hochschild complex, cannot not be extended to differential graded algebras.

The main tool is the Adams-Hilton model for a $CW$-complex whose 1-skeleton is trivial, [A-H]. To such a space $X$, they associate a differential graded chain algebra $A_X$ and a morphism

$$\theta : A_X \longrightarrow C_*(\Omega X, k),$$

which induces an isomorphism in homology.

From [B-L], there exists a free differential graded algebra $(T(V), d)$ and a morphism $\theta_0 : (T(V), d) \to A_X$, which induces an isomorphism in homology. Furthermore:

1. $V = \bigoplus_{n \geq 1} V_n, \quad V_n = H_{n+1}(X, k)$,
2. $d = d_2 + d_3 + \ldots + d_n + \ldots$ with $d_n(V) \subset V^{\otimes n}$ (we say that $(T(V), d)$ is minimal),
3. $d_2$ is dual to the multiplication on $H^*(X, k)$.

It is proved in [E-H] that $d_2 V \subset \mathbb{L}^2(V)$ where $\mathbb{L}(V)$ is the free Lie algebra generated by $V$.

A classical result of Goodwillie [Go] asserts that

$$HH_*(C_*(\Omega X)) = HH_*(T(V), d),$$
$$HC_*(C_*(\Omega X)) = HC_*(T(V), d)$$

as graded $k$-vector spaces.

Now, we give a similar result to theorem 1.3.

**Theorem 2.1.** *Let* $H = \bigoplus_{n \geq 0} H_n$ *be a finite dimensional graded* $k$*-vector space with* $H_0 = k$, $H_1 = 0$. *Let* $X$ *be any finite complex with* $H_*(X, k) = H$ *and denote by* $X_t$ *the wedge of spheres such that* $H_*(X_t, k) = H$. *Then we have*

1. $\dim H_n(X^{S^1}, k) \leq \dim H_n(X_t^{S^1}, k)$,
2. $\dim H_n^{S^1}(X^{S^1}, k) \leq \dim H_n^{S^1}(X_t^{S^1}, k)$

*for all* $n \geq 0$.

*Proof.* In [Vi2], we exhibit in theorem 1.5 (resp. theorem 2.4) a short complex that computes $HH_*(T(V), d)$ (resp. $HC_*(T(V), d)$). We use them here.

Recall that $HH_*(T(V), d) = H_*(T(V) \oplus T(V) \otimes \bar{V}, \delta)$ where

$$\bar{V} = sV, \quad (sV)_n = V_{n-1},$$
$$\delta_{|T(V)} = d$$

if $a \in T(V)$, $v \in \bar{V}$, $\delta(a \otimes \bar{v}) = (-1)^{|a|+|\bar{v}|}(Id - \tau)(a \otimes \bar{v}) + \tilde{d}(a \otimes \bar{v})$.

$I : T(V) \otimes \bar{V} \to T(V)$ satisfies $I(a \otimes \bar{v}) = av$,

$\tau : T(V) \otimes \bar{V} \to T(V)$ satisfies $\tau(a \otimes \bar{v}) = (-1)^{|a| \cdot |v|} va$,

$\tilde{d} : T(V) \otimes \bar{V} \to T(V) \otimes \bar{V}$ satisfies $\tilde{d}(a \otimes \bar{v}) = da \otimes \bar{v} + (-1)^{|a|} S(a, dv)$,

$S : T(V) \otimes T^+(V) \to T(V) \otimes \bar{V}$ satisfies

$$S(a \otimes v_1 \ldots v_p) = \sum_{i=1}^{p} \pm v_{i+1} \ldots v_p a v_1 \ldots v_{i-1} \otimes \bar{v}_i,$$

if $a \in A$, $v_i \in V$, and $|a|$ means the degree of $a$.

We define a bounded filtration on $(C_*, \delta) = (T(V) \oplus T(V) \otimes \bar{V}, \delta)$ by

$$\mathcal{F}_p(C_n) = C_n \quad \text{if} \quad p \geq n,$$
$$\mathcal{F}_{n-1}(C_n) = (T(V) \otimes \bar{V})_n,$$
$$\mathcal{F}_p(C_n) = 0 \quad \text{if} \quad p \leq n-2.$$

This determines a convergent spectral sequence $E^r_{pq}$ with

$$E^r_{pq} = 0 \quad \text{if} \quad q \neq 0 \quad \text{or} \quad q \neq 1,$$
$$E^1_{po} = [T(V)/Im(I-\tau)]_p,$$
$$E_{p-1,1} = [\mathrm{Ker}(I-\tau)]_p.$$

So we have, for a fixed $p \geq 0$,

$$E^1_{po} \oplus E^1_{p-1,1} = HH_p(T(V), 0) = H_p(X_t^{S^1}, k).$$

Since $E^r_{pq}$ converges to $HH_*(T(V), d) = H_*(X^{S^1}, k)$, part (1) is proved.

To prove part (2), we should recall the definition of the short complex $(\mathcal{B}_*, D)$ given in [Vi2] to compute $HC_*(T(V), d)$:

$$\mathcal{B}_* = k[u] \otimes C_* \quad \text{with} \quad |u| = 2,$$
$$D = 0 \quad \text{on} \quad k[u],$$
$$D = \delta = d \quad \text{on} \quad T(V),$$
$$D(u^n \otimes a \otimes \bar{v}) = u^n \otimes \delta(a \otimes \bar{v}) \quad \text{if} \quad a \in T(V), \bar{v} \in \bar{V}, n \in \mathbb{N},$$
$$D(u^n \otimes v_1 \dots v_m) = u^n \otimes d(v_1 \dots v_m) + u^{n-1} \otimes S(1, v_1 \dots v_m)$$

if $v_i \in V$, $1 \leq i \leq m$ and $n \geq 1$.

In fact, we will consider the reduced complex $\tilde{\mathcal{B}}_* = \mathcal{B}_*/k[u]$ with the induced differential $\tilde{D}$, that computes reduced cyclic homology

$$H\tilde{C}_*(T(V), d) := HC_*(T(V), d)/HC_*(k).$$

We define a bounded filtration on $(\tilde{\mathcal{B}}, \tilde{D})$ as follows:

$$\mathcal{F}_p(\tilde{\mathcal{B}}_n) = \tilde{\mathcal{B}}_n \quad \text{for} \quad p \geq n,$$
$$\mathcal{F}_{n-2q}(\tilde{\mathcal{B}}_n) = \bigoplus_{i \geq q} (T^+(V) \otimes u^i)_n \oplus \left( \bigoplus_{j \geq q} T(V) \otimes \bar{V} \otimes u^j \right)_n \quad \text{for} \quad q > 0,$$
$$\mathcal{F}_{n-2q-1}(\tilde{\mathcal{B}}_n) = \bigoplus_{i \geq q+1} (T^+(V) \otimes u^i)_n \oplus \left( \bigoplus_{j \geq q} T(V) \otimes \bar{V} \otimes u^j \right)_n \quad \text{for} \quad q \geq 0.$$

The induced spectral sequence $(E^r_{pq}, d^r)$ satisfies

$$E^0_{n-2q,2q} = (T^+(V) \otimes u^q)_n \quad \text{for} \quad q \geq 0,$$
$$E^0_{n-2q-1,2q+1} = (T(V) \otimes \bar{V} \otimes u^q)_n \quad \text{for} \quad q \geq 0.$$

The differential $d^0$ is precisely the differential which computes $H\tilde{C}_*(T(V), 0)$ using the complex of theorem 2.4 of [Vi2] with zero differential on $T(V)$. So we have

$$\bigoplus_{p+q=n} E^1_{pq} = H\tilde{C}_n(T(V), 0) = \tilde{H}^{S^1}_n(X_t^{S^1}, k).$$

Since $E^r_{pq}$ converges to $H\tilde{C}_*(T(V), d) = \tilde{H}^{S^1}_*(X^{S^1}, k)$, part (2) of theorem 2.1 is proved. ∎

**Example.** Considering $X = \mathbb{C}P^2$, we have $H^*(X,k) = k \oplus kx_2 \oplus kx_2^2$ with $|x_2| = 2$. We show, in [Vi2], that $\dim H_n(X^{S^1}) = 1$ for all $n \geq 1$. The corresponding space $X_t$ is $S^2 \bigvee S^4$ and we have shown in [Vi1] that the sequence $\dim H_n(X_t^{S^1})$ has exponential growth.

Now we are interested in spaces $X$ for which inequality (1) or (2) of theorem 2.1 becomes an equality.

**Theorem 2.2.** *Let $H = \bigoplus_{n\geq 0} H_n$ be a finite dimensional graded $k$-vector space with $H_0 = k$, $H_1 = 0$ and $N = \sup\{n | H_n \neq 0\}$ strictly positive. Let $X_t$ be the wedge of spheres such that $H_*(X_t,k) = H$. Let $X$ be a finite complex such that $H_*(X,k) = H$. We assume that either $\operatorname{char} k = 0$ or $\operatorname{char} k = p \geq 3N-3$, and*

$$\dim H_n^{S^1}(X^{S^1},k) = \dim H_n^{S^1}(X_t^{S^1},k) \quad \text{for all} \quad n \leq 3N-3.$$

*Then the multiplication is trivial on $\tilde{H}^*(X,k) = \bigoplus_{n>0} H^*(X,k)$.*

**Corollary 2.3.** *With the same hypothesis as in theorem 2.2, we assume furthermore that there exists an integer $r \geq 1$ such that $H_n = 0$ for $n \leq r$ and $n \geq 3r+2$. Then we have*

$$H_*(\Omega X,k) = H_*(\Omega X_t,k) = T(s^{-1}\bar{H})$$

*as graded algebras, where $\bar{H} = \bigoplus_{n>0} H_n$ and $(s^{-1}\bar{H})_n = H_{n+1}$.*

**Theorem 2.4.** *Let $H = \bigoplus_{n>0} H_n$ be a finite dimensional graded $k$-vector space with $H_0 = k$, $H_1 = 0$, and $N = \sup\{n | H_n \neq 0\}$ nonzero. Let $X_t$ be the wedge of spheres such that $H_*(X_t,k) = H$. Let $X$ be a finite CW-complex with $H_*(X,k) = H$. We assume that either $\operatorname{char} k = 0$, or $\operatorname{char} k = p > 3N-3$, and*

$$\dim H_n(X^{S^1},k) = \dim H_n(X_t^{S^1},k) \quad \text{for all} \quad n \leq 3N-3.$$

*Then the multiplication is trivial on $\tilde{H}^*(X,k) = \bigoplus_{n>0} H^*(X,k)$.*

**Corollary 2.5.** *With the same hypothesis as in theorem 2.4, we assume furthermore that there exists an integer $r \geq 1$ such that $H_n = 0$ for $n \leq r$ and $n \geq 3r+2$. Then we have*

$$H_*(\Omega X,k) = H_*(\Omega X_t,k) = T(s^{-1}\bar{H})$$

*as graded $k$-algebras.*

As in the algebraic context, the proof of theorem 2.4 is easy from theorem 2.2. We leave it to the reader.

*Proof of theorem 2.2.* From theorem 2.5 of [Vi2], we have

$$\tilde{H}_n^{S^1}(X_t^{S^1},k) = H\tilde{C}_n((T(V),0)) = \bigoplus_{q\geq 0}\bigoplus_{m\geq 1} H_{2q,n-2q}(\mathbb{Z}/m\mathbb{Z}, V^{\otimes m}) \oplus \left(\bigoplus_{q\geq 0}\bigoplus_{m\geq 1} H_{2q+1,n-2q-1}(\mathbb{Z}/m\mathbb{Z}, V^{\otimes m})\right).$$

This is exactly the term $\bigoplus_{p+q=n} E^1_{pq}$ in the spectral sequence defined on the complex $(\mathcal{B}_*, D)$ in the proof of theorem 2.1. If we assume that $\dim H^{S^1}_n(X^{S^1}_t, k) = \dim H^{S^1}_n(X^{S^1}, k)$, then the differentials $d_r$ on $\bigoplus_{p+q=n} E^r_{pq}$ should be zero, for all $r \geq 1$.

If we assume that $\operatorname{char} k = 0$, or $n \leq \operatorname{char} k$ if $\operatorname{char} k \neq 0$, then all the groups $H_{q,n-q}(\mathbb{Z}/m\mathbb{Z}, V^{\otimes m})$ are zero if $q \neq 0$. So we have

$$H\tilde{C}_n(T(V), 0) = E^1_{n,0} = \bigoplus_{m \geq 1} \left[ \frac{V^{\otimes m}}{Im(I - \tau_m)} \right]_n$$

and the differential $d^1 : E^1_{n,0} \to E^1_{n-1,0}$ is induced by the differential $d$ of $T(V)$.

Assume now that $H^{S^1}_n(X^{S^1}_t, k) = H^{S^1}_n(X, k)$ for all $n \leq 3N - 3$ (with the additional hypothesis $3N - 3 \leq \operatorname{char} k$ if $\operatorname{char} k \neq 0$).

Then we have necessarily

$$d(V^{\otimes m})_n \subset \bigoplus_{m \geq 2} [Im(I - \tau_m)]_{n-1}$$

for all $m \geq 1$, for all $n \leq 3N - 3$ if $\operatorname{char} k = 0$, or for all $n \leq 3N - 3 \leq \operatorname{char} k$ if $\operatorname{char} k \neq 0$.

We can write $d = d_2 + d_3 + \ldots + d_i + \ldots$ with $d_i(V) \subset V^{\otimes i}$ ; $i \geq 2$.

We will prove that $d_2 = 0$.

Notice that if $\operatorname{char} k = 3$, then the hypothesis of the theorem implies that $N = 2$; so we have $V = V_1$ and $d = 0$. In the following, we assume that $\operatorname{char} k > 3$. Suppose there exist elements $v \in V$ such that $d_2 v \neq 0$. Let $v$ be one of them of lowest degree $p$ for this property.

From [E-H], we can write

$$d_2 v = \sum_{i<j} \lambda_{ij} (u_i \otimes u_j - (-1)^{|u_i| \times |u_j|} u_j \otimes u_i) + \sum_i \mu_i u_i \otimes u_i$$

with $\lambda_{ij} \in k$, $\mu_i \in k$, and $\{u_i\}$ is a $k$-basis of $\bigoplus_{n<p} V_n$.

If $p$ is even, then $\mu_i = 0$, for all $i$.

If $p$ is odd, consider a term $\mu_i\, u_i \otimes u_i$ that appears in $d_2 v$; we have $v \otimes u_i \in (V^{\otimes 2})_n$ with $n \leq 3N - 3$, so that $d_2(v \otimes u_i)$ should belong to $Im(I - \tau_3)_{n-1} = \operatorname{Ker}(I + \tau_3 + \tau_3^2)_{n-1}$ since $\operatorname{char} k > 3$.

We have

$$\begin{aligned} d_2(v \otimes u_i) = d_2(v) \otimes u_i &= \sum_{k<j} \lambda_{kj} (u_k \otimes u_j \otimes u_i - (-1)^{|u_i| \times |u_j|} (u_j \otimes u_k \otimes u_i) \\ &+ \sum_{j \neq i} \mu_j u_j \otimes u_j \otimes u_i + \mu_i u_i \otimes u_i \otimes u_i, \end{aligned}$$

$$(I + \tau_3 + \tau_3^2)\,(d_2(v \otimes u_i)) = 3\mu_i\, u_i \otimes u_i \otimes u_i + X$$

where $X$ is a sum of monomials of $V^{\otimes 3}$, each of them containing some $u_j$ with $j \neq i$.

This implies $\mu_i = 0$ for all $i$.

So we have $d_2 v = \sum_{i<j} \lambda_{ij} (u_i \otimes u_j - (-1)^{|u_i| \times |u_j|} u_j \otimes u_i)$.

If $p$ is even, then $p-1$ is odd, so that necessarily, $|u_i|$ or $|u_j|$ is odd. Assume that $|u_i|$ is odd and consider

$$(I+\tau_3+\tau_3^2)\,(d_2(v\otimes u_i)) = -2\lambda_{ij}[u_i\otimes u_i\otimes u_j + u_j\otimes u_i\otimes u_i + u_i\otimes u_j\otimes u_i] + X$$

where $X$ belongs to $V^{\otimes 3}$, and $X$ is a sum of monomials, all containing some $u_k$, $k\neq i$, $k\neq j$.

Since $(I+\tau_3+\tau_3^2)\,(d_2(v\otimes u_i))$ should be zero, we have $\lambda_{ij}=0$. This proves that $d_2 v = 0$ if $|v|$ is even.

If $p$ is odd, we fix $i$ and $j$, $i<j$ and we consider $v\otimes u_i\otimes u_j$ that belongs to $(V^{\otimes 3})_n$ with $n\leq 3N-3$, so that $d_2(v\otimes u_i\otimes u_j) = (d_2 v)\otimes u_i\otimes u_j$ should belong to $Im(I-\tau_4)$, or equivalently to $\operatorname{Ker}(I+\tau_4+\tau_4^2+\tau_4^3) = K$ since $\operatorname{char} k = 0$, or $\operatorname{char} k > 3$.

We can suppose that $i=1$, $j=2$, so that we write

$$\begin{aligned} d_2(v\otimes u_1\otimes u_2) &= \lambda_{12}(u_1\otimes u_2\otimes u_1\otimes u_2 - (-1)^{|u_1|\times|u_2|}u_2\otimes u_1\otimes u_1\otimes u_2) \\ &\quad + \sum_{\substack{i<j\\(i,j)\neq(1,2)}} \lambda_{ij}(u_i\otimes u_j\otimes u_1\otimes u_2 - (-1)^{|u_j|\times|u_i|}u_j\otimes u_i\otimes u_1\otimes u_2). \end{aligned}$$

If $d_2(v\otimes u_1\otimes u_2)$ belongs to $K$, then we should have that

$$\lambda_{12}(u_1\otimes u_2\otimes u_1\otimes u_2 - (-1)^{|u_1|\times|u_2|}u_2\otimes u_1\otimes u_1\otimes u_2)$$

belongs to $K$.

This implies $(1+(-1)^{|u_1|+|u_2|})\lambda_{12}=0$, but $|u_1|+|u_2| = p-1$ is even, so that we should have $\lambda_{12}=0$. We have proved that

$$d_2 v = 0 \quad \text{if} \quad |v| \quad \text{is odd.}$$

Since $d_2$ is dual to the multiplication on $\tilde{H}^*(X,k) = \bigoplus_{n>0} H^n(X,k)$, [B-L], we have immediately that the multiplication is trivial on $\tilde{H}^*(X,k)$.

*Proofs of corollaries 2.3 and 2.5.* It is proved in [An] that spaces $X$ satisfying the additional hypothesis of the corollaries are $k$-formal; and for such spaces, the differential $d$ of their Adams-Hilton model is just equal to $d_2$. Since we have proved that the hypotheses of theorems 2.2 or 2.4 imply that $d_2=0$, in fact, for these spaces, $d=0$ in the Adams-Hilton model; so there is a morphism of differential graded algebras $\theta : (T(s^{-1}\bar{H}),0)\to C_*(\Omega X,k)$ that induces an isomorphism in homology; so we have

$$H_*(\Omega X,k) = T(s^{-1}H_*(X)).$$

*Remark.* More generally, corollary 2.5 means that if a space $X$ is $k$-formal and satisfies the hypothesis of theorem 2.4, then $H_*(\Omega X,k) = T(s^{-1}H_*(X))$ as algebras. If $k$ is the field of rational numbers, then $X$ has the same rational homotopy type as the wedge of spheres $X_t$. So, we can characterize wedge of spheres among other formal spaces with given homology as those having the largest Betti numbers for the free loop space.

If $\operatorname{char} k$ is positive, the situation is more complicated, since there exist formal spaces $X$ with $H_*(\Omega X,k) = T(s^{-1}\bar{X}_*(X))$, and the L.S. category of $X$ is two.

*Remark.* Consider the fibration $\Omega X\to X^I \xrightarrow{p} X\times X$ where $X^I = \{f:[0,1]\to X\}$ and $p(f) = (f(0),f(1))$. Let $\Delta$ be the diagonal map $X\to X\times X$ defined by

$\Delta(x) = (x, x)$. Then the free loop space $X^{S^1}$ is the total space of the pull-back of the fibration $p$. This gives rise to an Eilenberg-Moore spectral sequence in the second quadrant

$$E_2^{-pq} = \mathrm{Tor}_p^{H^*(X\times X)}(H^*(X,k), H^*(X,k))^q \Longrightarrow H^*(X^{S^1}, k).$$

If $X_f$ is a $k$-formal space with $H^*(X_f, k) = H^*(X, k)$ as graded algebras, the term $E_2$ is just $H^*(X_f^{S^1}, k)$; this implies that $\dim H^n(X^{S^1}, k) \leq \dim H^n(X_f^{S^1}, k)$, for all $n \geq 0$. We do not know examples of nonformal spaces $X$ such that $\dim H^n(X^{S^1}, k) = \dim H^n(X_f^{S^1}, k)$. In any case, it is easy to exhibit nonformal spaces $X$ such that

$$\dim H^n(\Omega X, \mathbb{Q}) = \dim H^n(\Omega X_f, \mathbb{Q}) \quad \text{for all} \quad n \geq 0.$$

It is sufficient to perturb the differential in the bigraded model of $H^*(X, \mathbb{Q})$ by adding a decomposable part [H-S]. So we conjecture that there exist nonformal spaces $X$ such that the Betti numbers of $X^{S^1}$ and $X_f^{S^1}$ are the same, and there is little hope to improve the conclusion of theorem 2.4.

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Université Paris-Nord, Institut Galilée, Département de Mathématiques, F-93430 Villetaneuse, France

*E-mail address*: vigue@math.univ-paris13.fr